CR Embeddings, Chains, and the Fefferman Bundle

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Introduction

Suppose $f : M \hookrightarrow \hat{M}$ is an embedding of a CR manifold M into a CR manifold $\hat{M}$ of strictly larger dimension with both M and $\hat{M}$ strictly pseudoconvex and of hypersurface type. In Riemannian geometry the Levi-Civita connection induces a family of curves called geodesics defined by a certain second order partial differential equation. On a CR manifold chains are a special type of CR invariant curve defined by a second order partial differential equation determined by the CR structure. Chains determine the CR structure of a manifold in the sense of the result by J. Cheng in 1988 ([4]) which says that any (local) diffeomorphism between two nondegenerate CR manifolds which preserves chains must in fact be a CR (or conjugate CR) diffeomorphism. In the language of the famous paper by S. S. Chern and J. K. Moser ([1]) a curve in a strictly pseudoconvex CR manifold M is a chain if there exists a complete system of forms on the structure bundle over M $\{\omega, \omega^{\alpha}, \omega^{\bar{\beta}}, \phi, \phi_{\alpha}{}^{\beta}, \phi^{\alpha}, \phi^{\bar{\beta}}, \psi\}$ such that, after pulling back to M, along the curve we have:

$$\theta^{\alpha} = \phi^{\alpha} = 0$$

In the Riemannian setting, vanishing of the second fundamental form of a smooth embedding is equivalent to the preservation of geodesics. In the CR setting this may not generally be the case, however we will see below that, in the case that the target is spherical (ie $\hat{M} \cong_{loc} \mathbb{S}$), a CR embedding

1

$f : M \rightarrow \mathbb{S}$ preserves chains if and only if it's CR second fundamental form vanishes. Moreover in this paper we will discover a different geometric property which is equivalent to the above conditions, in the case that the target is locally spherical.

In his 1976 paper "'Monge-Ampère equations, the Bergman kernel, and geometry of pseudoconvex domains,"' ([5]) C. L. Fefferman developed a circle bundle over the boundary of a strictly pseudoconvex domain in $\mathbb{C}^n$ equipped with a Lorentz metric that carried with it information about the CR structure of the boundary. His method involved using an approximation of a solution to the Monge-Ampère equations. In particular he proves that chains are the projections of light rays (aka null geodesics) on the circle bundle down to the boundary of the domain. In 1977 D. Burns, Jr., K. Diederich, and S. Shnider ([6]) and S. M. Webster ([7]) introduced intrinsic constructions of the Fefferman metric thereby generalizing the Fefferman bundle to abstract CR manifolds with nondegenerate Levi Form (positive signature is allowed). The conformal class of the Fefferman metric is a CR invariant which we will denote by [h] where h is a metric representing the conformal class. Let the collection (C → M, [h]) be called the Fefferman bundle. Now we can ask the following question:

> Under what conditions may the embedding f be lifted to a conformal isometric embedding of C into $\hat{C}$?

It is easy to see that any CR diffeomorphism between two equidimensional CR hypersurfaces locally lifts to a conformal isometry. If the target is higher dimensional we will show that a necessary and sufficient condition for such an isometric lift to exist will be a certain relationship between the conformal curvature tensor of the ambient space $\hat{M}$, the second fundamental form of the embedding f, and the CR dimension of M (see Theorem 4.1). We will also see that this condition is implied by the preservation of chains condition. In the case that the ambient space $\hat{M}$ is locally spherical the equations reduce to a particularly simple result.

Theorem 0.1 *Suppose $f : M \hookrightarrow S^{2\hat{n}+1}$ is a CR embedding of a strictly pseudoconvex hypersurface M into a sphere of larger dimension. The following*

conditions are equivalent.

1. *f preserves chains*
2. *There is a local lift of f to a conformal isometry of Fefferman metrics*
3. *The CR second fundamental form of f vanishes*
4. *There exists a local CR diffeomorphism ϕ from the sphere $\mathbb{S}^{2n+1}$ to M and an automorphism of the target sphere $A \in Aut(\mathbb{S}^{2\hat{n}+1})$ such that the composition $A \circ f \circ \phi : \mathbb{S}^{2n+1} \to \mathbb{S}^{2\hat{n}+1}$ is the linear embedding.*

We say that M admits a (pseudohermitian) pseudo-Einstein structure if it admits a contact form θ so that the associated pseudohermitian Ricci and scalar curvatures satisfy:

$$R_{\alpha\bar{\beta}} - \frac{1}{2(n+1)} R g_{\alpha\bar{\beta}} \equiv 0$$

In the process we will establish the following (local) implication in the case of a general target space $\hat{M}$:

Proposition 1 *If M and $\hat{M}$ admit an adapted pseudo-Einstein structure with respect to the embedding $f : M \to \hat{M}$ (that is they admit an adapted coframe which is pseudo-Einstein for both M and $\hat{M}$) then f may be lifted to a conformal isometry of the associated Fefferman metrics.*

1 Adapted Frames

Let M be a strictly pseudoconvex CR manifold of hypersurface type. There is a subbundle of the tangent space of M, H, called the complex tangent space of M. A choice of a nonvanishing real 1-form $\theta \in H^{\perp}$ which annihilates H is called a contact form on M. Fixing such a choice of θ, the pair (M, θ) is called a choice of pseudohermitian structure. We say $\{\theta, \theta^{\alpha}, \theta^{\bar{\alpha}}\}$ is an admissible coframe for M if the Levi form is given as the identity matrix; $d\theta = ig_{\alpha\bar{\beta}}\theta^{\alpha} \wedge \theta^{\bar{\beta}}$ where $g_{\alpha\bar{\beta}} = \delta_{\alpha\bar{\beta}}$ and $1 \leq \alpha \leq n$ where $\dim M = 2n+1$.
In [8] Webster shows that associated to a pseudohermitian structure on M is a family of connection 1-forms $\omega_{\alpha}{}^{\beta}$ which are uniquely determined by the choice of frame $\{\theta, \theta^{\alpha}\}$ and the conditions:

$$d\theta^{\beta} = \theta^{\alpha} \wedge \omega_{\alpha}{}^{\beta} + \theta \wedge \tau^{\beta} \tag{1.1}$$

$$dg_{\alpha\bar{\beta}} = \omega_{\alpha\bar{\beta}} + \omega_{\bar{\beta}\alpha} \tag{1.2}$$

with $\tau^\beta = A^\beta{}_{\bar{\mu}}\theta^{\bar{\mu}}$, $A^{\alpha\beta} = A^{\beta\alpha}$, and here we are using the summation convention with the matrix $(g_{\alpha\bar{\beta}})$ to raise an lower indices, (e.g. $\omega_{\alpha\bar{\beta}} = \omega_\alpha{}^\gamma g_{\gamma\bar{\beta}}$). Webster showed that these forms determine a unique connection on H. The analogous forms on $\hat{M}$ will be given a hat. He then showed that the pseudohermitian curvature $R_\alpha{}^\beta{}_{\mu\bar{\nu}}$ satisfies

$$d\omega_\alpha{}^\beta - \omega_\alpha{}^\gamma \wedge \omega_\gamma{}^\beta = R_\alpha{}^\beta{}_{\mu\bar{\nu}}\theta^\mu \wedge \theta^{\bar{\nu}} + W_\alpha{}^\beta{}_\mu\theta^\mu \wedge \theta - W^\beta{}_{\alpha\bar{\nu}}\theta^{\bar{\nu}} \wedge \theta + i\theta_\alpha \wedge \tau^\beta - \tau_\alpha \wedge \theta^\beta \tag{1.3}$$

Given the admissible coframe for M as above it is shown in [3] that locally there is a pseudohermitian structure $(\hat{M}, \hat{\theta})$ on $\hat{M}$ and an admissible coframe $\{\hat{\theta}, \hat{\theta}^A\}$ for $\hat{M}$, where $1 \leq A \leq \hat{n}$, such that when pulled back to M via the embedding $f : M \hookrightarrow \hat{M}$ we have

$$\begin{aligned} \hat{\theta} &= \theta \\ \hat{\theta}^\alpha &= \theta^\alpha \\ \hat{\theta}^a &= 0 \end{aligned} \tag{1.4}$$

where we use lower case Roman letters to denote the normal direction, $n+1 \leq a \leq \hat{n}$. Such a pair of frames is known as an adapted pair of coframes.

It is shown in [3] that on M we have $\hat{\omega}_\alpha{}^\beta = \omega_\alpha{}^\beta$, $\hat{\tau}^\alpha = \tau^\alpha$, $\tau^b = 0$, $\omega_\alpha{}^b = \omega_\alpha{}^b{}_\beta\theta^\beta$, $\omega_\alpha{}^b{}_\beta = \omega_\beta{}^b{}_\alpha$, and that the CR second fundamental form Π is then given by:

$$\Pi(L_\alpha, L_\beta) = \omega_\alpha{}^b{}_\beta L_b \tag{1.5}$$

2 The Fefferman Bundle

We now fix a choice of pseudohermitian structure θ and an admissible coframe $\{\theta, \theta^\alpha, \theta^{\bar{\alpha}}\}$ on M. Let $C = M \times S^1$ and locally define the 1-form σ by:

$$\sigma = \frac{1}{n+2}(dt + i\omega_\alpha{}^\alpha - \frac{1}{2(n+1)}R\theta - \frac{i}{2}g^{\alpha\bar{\beta}}dg_{\alpha\bar{\beta}}) \tag{2.1}$$

where the variable t parameterizes the S^1 coordinate over M. Then we define the metric h on C by:

$$h = \theta^\alpha \cdot \theta_\alpha + 2\theta \cdot \sigma \tag{2.2}$$

4

Here the '·' means symmetric product. It is shown in [9] that both the form σ and the conformal class of the Lorentz metric h are independent of the choice of admissible coframe $\{\theta^\alpha\}$, are globally defined on C, and that h corresponds to the Fefferman metric previously developed by Fefferman (among others [5], [6]). In [7] it is shown that the projections of null geodesic from C onto M are chains (excluding the fibers of C which project to points) and all chains are given as the projection of a null geodesic in C.

Lemma 2.1 *With respect to this frame we have:*

$$R_{\alpha\bar\beta} - \frac{1}{2(n+1)}Rg_{\alpha\bar\beta} = 0 \quad \Longleftrightarrow \quad i\omega_\alpha{}^\alpha - \frac{1}{2(n+1)}R\theta \quad \text{is closed} \quad (2.3)$$

Note: The condition on the left hand side of this equivalence is to say that θ is pseudo-Einstein.
The proof of this lemma appears in [10] and will be given here out of interest. It requires the following useful lemma involving the complex tangent space H of M.

Lemma 2.2 *If ξ is a closed 2-form on M such that $\xi|H = 0$ then $\xi \equiv 0$.*

Given a choice of contact form θ we have $H^\perp = \langle\theta\rangle$. The condition $\xi|H = 0$ implies that $\xi = \eta \wedge \theta$ for some 1-form η. The assumption that ξ is closed then gives $0 = d\eta \wedge \theta + \eta \wedge d\theta$. Restricting to H we then see $\eta \wedge d\theta|_H = 0$. Since the Levi form is nondegenerate we then have $\eta|_H = 0$ (ie $\eta \equiv 0 \mod\theta$). Thus $\xi = \eta \wedge \theta = 0$. QED

Now to prove Lemma 2.1 we observe from equation 1.3 and by the symmetry properties of the connection forms that we have:

$$d\omega_\alpha{}^\alpha = d\omega_\alpha{}^\alpha - \omega_\alpha{}^\gamma \wedge \omega_\gamma{}^\alpha \equiv R_{\mu\bar\nu}\theta^\mu \wedge \theta^{\bar\nu} \mod \theta \quad (2.4)$$

Thus we see $R_{\alpha\bar\beta} - \frac{1}{2(n+1)}Rg_{\alpha\bar\beta} = 0$ if and only if

$$d\omega_\alpha{}^\alpha \equiv -\frac{i}{2(n+1)}Rd\theta \equiv -\frac{i}{2(n+1)}d(R\theta) \mod \theta \quad (2.5)$$

which, by lemma 2.2, holds if and only if

$$d\omega_\alpha{}^\alpha + \frac{i}{2(n+1)}d(R\theta) \equiv 0 \quad (2.6)$$

QED

5

3 Chains

Here we will discuss the definitions of chains. The definitions in this section are independent of the choice of pseudohermitian structure. A choice of admissible coframe $\{\theta, \theta^\alpha, \theta^{\bar\alpha}\}$ (where $\theta \in H^\perp, \theta \neq 0$ is a real 1-form) induces a family of 1-forms on M $\{\phi, \phi_\beta{}^\alpha, \phi^\alpha, \phi^{\bar\alpha}, \psi\}$ which are the pullbacks of a complete system of forms on a principle G-bundle over M ([1]). A curve γ which is transverse to the CR tangent space (ie $\theta(\dot\gamma) \neq 0$ for any choice of $\theta \in H^\perp$) is a chain if when we choose our admissible frame θ^α so that along γ

$$\theta^\alpha = 0 \tag{3.1}$$

then we also have,

$$\phi^\alpha = 0 \tag{3.2}$$

It is not hard to show that this definition is independent of the choice of θ^α satisfying condition (3.1).

The above definition can be very complicated to work with because, for a given curve, one must choose a coframe associated to that curve satisfying 3.1. When dealing with more than one chain a geometric comparison is then very difficult. In the proof of [4] another formulation of the definition of a chain is given which may be used with any given admissible coframe.

Lemma 3.1 *Suppose M is a strictly pseudoconvex CR manifold of hypersurface type. A curve γ in M is a chain if and only if for any admissible coframe $\{\theta, \theta^\alpha, \theta^{\bar\alpha}\}$ the following equation has a solution $a(t) = (a^1, ..., a^n)$ along γ:*

$$\begin{cases} \theta^\alpha = 2a^\alpha\theta \\ da^\alpha = 4ia^\alpha|a|^2\theta - a^\alpha(\pi_0{}^0 + \bar\pi_0{}^0) - \phi_\beta{}^\alpha a^\beta - \frac{1}{2}\phi^\alpha \end{cases} \tag{3.3}$$

where $\pi_0{}^0 = -\frac{1}{n+2}(\phi_\alpha{}^\alpha + \phi)$.

Note: When we choose an admissible coframe we have $\phi = 0$ on M. Moreover by the anti-symmetry property of the forms $\phi_\alpha{}^\beta$ (see the appendix of [1]) we conclude $\pi_0{}^0 + \bar\pi_0{}^0 = 0$ and thus equation 3.3 may be written:

$$\begin{cases} \theta^\alpha = 2a^\alpha\theta \\ da^\alpha = 4ia^\alpha|a|^2\theta - \phi_\beta{}^\alpha a^\beta - \frac{1}{2}\phi^\alpha \end{cases} \tag{3.4}$$

Proof of lemma 3.1: Fix a chain $\gamma \subset M$. Fix a contact form θ on M and an admissible coframe $\{\theta, \theta^\alpha\}$. The admissibility condition means

$$d\theta = i g_{\alpha\bar{\beta}} \theta^\alpha \wedge \theta^{\bar{\beta}}$$

were the $n \times n$ matrix $g_{\alpha\bar{\beta}}$ is the identity matrix representing the Levi Form. The coframe constructed in [1] may be written in the form of a Maurer-Cartan form as:

$$\pi = \begin{pmatrix} \pi_0{}^0 & \theta^\alpha & 2\theta \\ -i\phi_\alpha & (\phi_\alpha{}^\beta + \pi_0{}^0 \delta_\alpha{}^\beta) & 2i\theta_\alpha \\ -\frac{1}{4}\psi & \frac{1}{2}\phi^\alpha & -\bar{\pi}_0{}^0 \end{pmatrix} \tag{3.5}$$

This family of forms is unique up to transformation by an element of the G-structure, $h \in G$. The transformation is then given on M by:

$$\tilde{\pi} = dh h^{-1} + h\pi h^{-1} \tag{3.6}$$

We are free to choose a frame θ, θ^α so that along γ we have $\theta \equiv 1$. It is shown in [6] that the element $h \in G$ may be chosen to be of the form:

$$h(t) = \begin{pmatrix} 1 & 0 & 0 \\ -2ia^* & I & 0 \\ -i|a|^2 & a & 1 \end{pmatrix} , a(t) \in \mathbb{C}^n \tag{3.7}$$

γ is a chain if and only if there is some transformation h so that $\tilde{\theta}^\alpha = 0 = \tilde{\phi}^\alpha$. One may now isolate the $\tilde{\theta}^\alpha$, $\tilde{\phi}^\alpha$ terms from equation (3.6) and set them equal to 0. Using:

$$h^{-1}(t) = \begin{pmatrix} 1 & 0 & 0 \\ 2ia^* & I & 0 \\ -i|a|^2 & -a & 1 \end{pmatrix} , a(t) \in \mathbb{C}^n \tag{3.8}$$

and letting $I, J \in 0, ..., n+1$ we have:

$$2\tilde{\theta} = \tilde{\pi}_0{}^{n+1} = (dh)_0{}^I (h^{-1})_I{}^{n+1} + h_0{}^I \pi_I{}^J (h^{-1})_J{}^{n+1}$$
$$= 2\theta$$

$$\tilde{\theta}^\alpha = \tilde{\pi}_0{}^\alpha = (dh)_0{}^I (h^{-1})_I{}^\alpha + h_0{}^I \pi_I{}^J (h^{-1})_J{}^\alpha$$
$$= (\phi_0{}^0, \theta^\beta, 2\theta) \bullet (0, \delta_\beta{}^\alpha, -a^\alpha)$$
$$= \theta^\alpha - 2a^\alpha \theta$$

7

$$\frac{1}{2}\tilde{\phi}^\alpha = \tilde{\pi}_{n+1}{}^\alpha = (dh)_{n+1}{}^I(h^{-1})_I{}^\alpha + h_{n+1}{}^I\pi_I{}^J(h^{-1})_J{}^\alpha$$

$$= da^\alpha + (-i|a|^2, a^\beta, 1) \bullet (\theta^\alpha - 2a^\alpha, \phi_\beta{}^\alpha + \pi_0{}^0\delta_\beta{}^\alpha - 2ia^\alpha\theta_\beta, \frac{1}{2}\phi^\alpha + a^\alpha\bar{\pi}_0{}^0)$$

$$= da^\alpha + a^\beta\phi_\beta{}^\alpha + a^\beta\pi_0{}^0\delta_\beta{}^\alpha - 2ia^\alpha a^\beta\theta_\beta + \frac{1}{2}\phi^\alpha + a^\alpha\bar{\pi}_0{}^0$$

$$= da^\alpha - 2ia^\alpha a^\beta\theta_\beta + a^\beta\phi_\beta{}^\alpha + a^\alpha(\pi_0{}^0 + \bar{\pi}_0{}^0) + \frac{1}{2}\phi^\alpha$$

Setting $\theta^\alpha = \phi^\alpha = 0$ results in the desired differential equation on $a(t)$.

4 An Isometric Lift

We will see that the existence of an adapted pseudo-Einstein pair of coframes (that is an adapted pair such that each coframe is pseudo-Einstein) is a sufficient but not a necessary condition on an adapted coframe of the embedding $f : M \hookrightarrow \hat{M}$ so that f may be lifted to a conformal isometry. The goal of this section is to prove the following theorem:

Theorem 4.1 *The map $f : M \hookrightarrow \hat{M}$ may be locally lifted to a conformal isometry between Fefferman metrics if and only if the following equation holds with respect to any adapted frame.*

$$\hat{S}_a{}^a{}_{\alpha\bar{\beta}} + \omega_\mu{}^a{}_\alpha\omega^\mu{}_{a\bar{\beta}} = \frac{1}{2(n+1)}(\hat{S}_a{}^a{}_\mu{}^\mu + \omega_\mu{}^a{}_\nu\omega^\mu{}_a{}^\nu)g_{\alpha\bar{\beta}} \qquad (4.1)$$

Equation 4.1 is a condition on part of the CR conformal curvature tensor $\hat{S}$ of $\hat{M}$, the second fundamental form of the map f, and the dimension of M. Thus it is a condition on the CR structures of M and $\hat{M}$ independent of the choice of pseudohermitian structure.

Let $\{\hat{\theta}, \hat{\theta}^A, \hat{\theta}^A\}$ be a frame on $\hat{M}$ adapted to the frame $\{\theta, \theta^\alpha, \theta^\alpha\}$ on M. Recall by equation 2.2 the conformal classes of the Fefferman metrics on C and $\hat{C}$ are represented by the metrics $h = \theta^\alpha \cdot \theta_\alpha + 2\theta \cdot \sigma$ and $\hat{h} = \hat{\theta}^A \cdot \hat{\theta}_A + 2\hat{\theta} \cdot \hat{\sigma}$ respectively. With respect to this adapted frame, since $f^*(\hat{\theta}^A \cdot \hat{\theta}_A) = \theta^\alpha \cdot \theta_\alpha$ and $f^*\hat{\theta} = \theta$, any lift $F : C \hookrightarrow \hat{C}$ of the CR embedding $f : M \hookrightarrow \hat{M}$ is a conformal isometry if and only if it is actually an isometry and $f^*\hat{\sigma} = \sigma$.

By [3] a relationship between Webster's connection 1-forms and the pull backs of the Chern-Moser forms is given by:

$$\phi_\beta{}^\alpha = \omega_\beta{}^\alpha + D_\beta{}^\alpha\theta, \qquad \phi^\alpha = \tau^\alpha + D_\mu{}^\alpha\theta^\mu + E^\alpha\theta \qquad (4.2)$$

where

$$D_{\alpha\bar\beta} = \frac{i}{n+2}R_{\alpha\bar\beta} - \frac{i}{2(n+1)(n+2)}Rg_{\alpha\bar\beta}$$

$$E^\alpha = \frac{2i}{2n+1}(A^{\alpha\mu}{}_{;\mu} - D^{\bar\nu\alpha}{}_{;\bar\nu})$$

(4.3)

Here we see that the pseudo-Einstein condition is equivalent to $D_{\alpha\bar\beta} = 0$. The pullbacks of the associated forms on $\hat M$ are related by:

$$\hat\phi_\beta{}^\alpha = \phi_\beta{}^\alpha + C_\beta{}^\alpha\theta, \qquad \hat\phi^\alpha = \phi^\alpha + C_\mu{}^\alpha\theta^\mu + F^\alpha\theta$$

(4.4)

where

$$C_\beta{}^\alpha = \hat D_\beta{}^\alpha - D_\beta{}^\alpha, \qquad F^\alpha = \hat E^\alpha - E^\alpha$$

(4.5)

It is shown in [3] using the pseudohermitian Gauss equation that:

$$C_{\alpha\bar\beta} = \frac{i}{n+2}(\hat S_a{}^a{}_{\alpha\bar\beta} + \omega_\mu{}^a{}_\alpha\omega^\mu{}_{a\bar\beta} - \frac{1}{2(n+1)}(\hat S_a{}^{a\mu}{}_\mu + \omega_\mu{}^a{}_\nu\omega^\mu{}_a{}^\nu)g_{\alpha\bar\beta})$$

(4.6)

We will take the variable s for the S^1 coordinate in $\hat C$. Using equation (3.1) the condition $f^*\hat\sigma = \sigma$ then becomes:

$$\frac{1}{\hat n + 2}ds = \frac{1}{n+2}dt + \frac{1}{n+2}(i\omega_\alpha{}^\alpha - \frac{1}{2(n+1)}R\theta) - \frac{1}{\hat n+2}(i\hat\omega_A{}^A - \frac{1}{2(\hat n+1)}\hat R\theta)$$

(4.7)

This is a differential equation in $s = s(x,t)$ (where x denotes local coordinates on M). Locally a form is closed if and only if it is exact, thus we may locally solve equation 4.7 if and only if the term on the right hand side is closed. We observe,

$$\frac{1}{n+2}(i\omega_\alpha{}^\alpha - \frac{1}{2(n+1)}R\theta) - \frac{1}{\hat n+2}(i\hat\omega_A{}^A - \frac{1}{2(\hat n+1)}\hat R\theta) \text{ is closed}$$

$$\Leftrightarrow \quad d(\frac{1}{n+2}\omega_\alpha{}^\alpha - \frac{1}{\hat n+2}\hat\omega_A{}^A) \equiv -i(\frac{1}{2(n+1)(n+2)}R - \frac{1}{2(\hat n+1)(\hat n+2)}\hat R)d\theta \text{ mod } \theta$$

(by Lemma 2.2)

$$\Leftrightarrow \quad (\frac{1}{n+2}R_{\alpha\bar\beta} - \frac{1}{\hat n+2}\hat R_{\alpha\bar\beta}) = -(\frac{1}{2(n+1)(n+2)}R - \frac{1}{2(\hat n+1)(\hat n+2)}\hat R)g_{\alpha\bar\beta}$$

(by equation 2.4 and since $d\theta = ig_{\alpha\bar\beta}\theta^\alpha \wedge \theta^{\bar\beta}$)

$$\Leftrightarrow \quad C_{\alpha\bar\beta} = \hat D_{\alpha\bar\beta} - D_{\alpha\bar\beta} = 0$$

(4.8)

9

This means that an isometric lift of f exists if and only if we have $\hat{\phi}_\beta{}^\alpha = \phi_\beta{}^\alpha$ with respect to any adapted frame. By 4.6 this is equivalent to:

$$\hat{S}_a{}^a{}_{\alpha\bar{\beta}} + \omega_\mu{}^a{}_\alpha \omega^\mu{}_{a\bar{\beta}} = \frac{1}{2(n+1)}(\hat{S}_a{}^a{}_\mu{}^\mu + \omega_\mu{}^a{}_\nu \omega^\mu{}_a{}^\nu)g_{\alpha\bar{\beta}} \qquad (4.9)$$

QED

Let an adapted frame be called an adapted pseudo-Einstein structure if both coframes on M and $\hat{M}$ are pseudo-Einstein structures. It is clear by Lemma 2.1 and equation 4.8 that if the embedding f admits an adapted pseudo-Einstein structure then f may be lifted to an conformal isometry. Thus we have established Proposition 1 from the introduction of this paper.

5   Comparing the Second Fundamental Forms

It is no surprise that the second fundamental forms of f and a conformal isometric lift F are related in a simple way. Fix an adapted frame as before. We will need this technical lemma:

Lemma 5.1

$$\nabla \omega_\alpha{}^a{}_\gamma \equiv -\hat{R}_\alpha{}^a{}_{\gamma\bar{\nu}}\theta^{\bar{\nu}} \ mod\theta, \ \theta^\beta \qquad (5.1)$$

To prove this lemma we start with the following identity from [3]:

$$\nabla \omega_\alpha{}^a{}_\gamma = d\omega_\alpha{}^a{}_\gamma - \omega_\mu{}^a{}_\gamma \omega_\alpha{}^\mu + \omega_\alpha{}^b{}_\gamma \omega_b{}^a - \omega_\alpha{}^a{}_\mu \omega_\gamma{}^\mu \qquad (5.2)$$

Working $mod\theta$, pulling back to M, and using the $\hat{M}$ analog of equations (1.3), (1.5), (1), (5.2) we have:

$$\begin{aligned}
-\hat{R}_\alpha{}^a{}_{\mu\bar{\nu}}\theta^\mu \wedge \theta^{\bar{\nu}} &\equiv \omega_\alpha{}^\gamma \wedge \omega_\gamma{}^a + \omega_\alpha{}^b \wedge \omega_b{}^a - d\omega_\alpha{}^a \\
&\equiv \omega_\alpha{}^\gamma \wedge \omega_\gamma{}^a{}_\beta \theta^\beta + \omega_\alpha{}^b{}_\beta \theta^\beta \wedge \omega_b{}^a - d(\omega_\alpha{}^a{}_\beta \theta^\beta) \\
&\equiv \theta^\beta \wedge (\omega_\alpha{}^b{}_\beta \omega_b{}^a - \omega_\gamma{}^\alpha{}_\beta \omega_\alpha{}^\gamma - \omega_\alpha{}^a{}_\gamma \omega_\beta{}^\gamma + d\omega_\alpha{}^a{}_\beta) \\
&\equiv \theta^\beta \wedge \nabla \omega_\alpha{}^a{}_\beta \ \mathrm{mod} \ \theta
\end{aligned}$$

This ends the proof of Lemma 5.1 . QED

Using the coframe $\{\theta, \theta^\alpha, \theta^{\bar{\alpha}}\}$ on M we may take $\{\theta, \sigma, \theta^\alpha, \theta^{\bar{\alpha}}\}$ as a coframe

on C and $\{T, X, L_\alpha, L_{\bar{\alpha}}\}$ as a dual frame. The metric h is then given by:

$$h = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}g_{\alpha\bar{\beta}} \\ 0 & 0 & \frac{1}{2}g_{\bar{\alpha}\beta} & 0 \end{pmatrix} \tag{5.3}$$

The following lemma is proved in [9]:

Lemma 5.2 *The Levi-Civita connection 1-forms of h are given by:*

$$\Omega = \begin{pmatrix} 0 & 0 & i\sigma^\alpha & -i\sigma^{\bar{\alpha}} \\ 0 & 0 & i\theta^\alpha & -i\theta^{\bar{\alpha}} \\ \frac{i}{2}\theta_\beta & \frac{i}{2}\sigma_\beta & \sigma_\beta{}^\alpha & 0 \\ -\frac{i}{2}\theta_{\bar{\beta}} & -\frac{i}{2}\sigma_{\bar{\beta}} & 0 & \sigma_{\bar{\beta}}{}^{\bar{\alpha}} \end{pmatrix} \tag{5.4}$$

where

$$\sigma_\beta{}^\alpha = \omega_\beta{}^\alpha + D_\beta{}^\alpha\theta + i\delta_\beta{}^\alpha\sigma = \phi_\beta{}^\alpha + i\delta_\beta{}^\alpha\sigma$$

$$\sigma_\beta = i\tau_\beta + D_{\beta\bar{\gamma}}\theta^{\bar{\gamma}} + C_\beta\theta, \quad C_\beta = \frac{2}{n+2}(A_{\alpha\beta;}{}^\alpha + \frac{i}{2(n+1)}R_{;}{}^a)$$

The analogous procedure using the coframe $\{\hat{\theta}, \hat{\sigma}, \hat{\theta}^A, \hat{\theta}^{\bar{A}}\}$ on $\hat{C}$ gives:

$$\hat{\Omega} = \begin{pmatrix} 0 & 0 & i\hat{\sigma}^A & -i\hat{\sigma}^{\bar{A}} \\ 0 & 0 & i\hat{\theta}^A & -i\hat{\theta}^{\bar{A}} \\ \frac{i}{2}\hat{\theta}_B & \frac{i}{2}\hat{\sigma}_B & \hat{\sigma}_B{}^A & 0 \\ -\frac{i}{2}\hat{\theta}_{\bar{B}} & -\frac{i}{2}\hat{\sigma}_{\bar{B}} & 0 & \hat{\sigma}_{\bar{B}}{}^{\bar{A}} \end{pmatrix} \tag{5.5}$$

Since $F^*\hat{\sigma} = \sigma$ the second fundamental form is then given by the pull back via F of the forms:

$$\{\hat{\Omega}_\alpha{}^a\} = \{i\hat{\sigma}^a, \ i\theta^a, \ -i\hat{\sigma}^{\bar{a}}, \ -i\theta^{\bar{a}}, \ \hat{\sigma}_\alpha{}^a, \ \hat{\sigma}_{\bar{\alpha}}{}^{\bar{a}}\} \tag{5.6}$$

On C we have $\tau^a = 0 = \theta^a$ and thus by equation 1, $A_\alpha{}^a \equiv 0$. It is also clear that $\hat{D}^a{}_\gamma = \frac{i}{\hat{n}+2}\hat{R}^a{}_\gamma$. Thus the non-trivial terms in the second fundamental of F are

$$\left\{ \begin{array}{c} \frac{i}{\hat{n}+2}\hat{R}^a{}_\gamma\theta^\gamma - \frac{1}{(\hat{n}+1)(\hat{n}+2)}\hat{R}_{;}{}^a\theta \\ \\ \hat{\omega}_\alpha{}^a + \frac{i}{\hat{n}+2}\hat{R}_\alpha{}^a\theta \end{array} \right\} \tag{5.7}$$

and their complex conjugates. We are now in a position to prove the following lemma.

11

Lemma 5.3

(1) *If the second fundamental form of F vanishes the second fundamental form of f vanishes.*

(2) *If the second fundamental form of f vanishes then the second fundamental form of F vanishes if and only if $\hat{R}_{;}{}^a \equiv 0$ with respect to any adapted frame.*

First assume the second fundamental form of F vanishes. Then all the forms in 5.7 vanish. The vanishing of $\frac{i}{\hat{n}+2}\hat{R}^a{}_\gamma \theta^\gamma + iC^a\theta$ implies that $\hat{R}^a{}_\gamma \equiv 0$. Combining this with the vanishing of $\hat{\omega}_\alpha{}^a + \frac{i}{\hat{n}+2}\hat{R}_\alpha{}^a\theta$ yields $\hat{\omega}_\alpha{}^a = 0$
Now let us assume $\hat{\omega}_\alpha{}^a = 0$. By lemma 5.1 we have $\hat{R}_\alpha{}^a \equiv 0$. Thus we see that all terms in equation 5.7 vanish except $\hat{R}_{;}{}^a \equiv 0$. QED.

6 Chain Preserving CR Embeddings

We will prove the following.

Theorem 6.1 *A chain preserving CR embedding of strictly pseudoconvex CR hypersurfaces $f : M \to \hat{M}$ may be locally lifted to a conformal isometry of the associated Fefferman metrics.*

We consider a pair of coframes (θ, θ^α), $(\hat{\theta}, \hat{\theta}^A)$, adapted with respect to the embedding f, defined near a point $p \in M$ and $f(p) \in \hat{M}$ respectively. Fix an arbitrary chain $\gamma \subset M$ through p. By assumption $f(\gamma)$ is a chain through $f(p)$ in $\hat{M}$. Parameterize γ so that $\gamma(0) = p$. There exists functions $a^\alpha(t)$, $\hat{a}^A(t)$ so that

$$\begin{aligned}\theta^\alpha &= 2a^\alpha(t)\theta \\ \hat{\theta}^A &= 2\hat{a}^A(t)\hat{\theta}\end{aligned} \tag{6.1}$$

along γ and $f \circ \gamma$ respectively and thus since these curves are chains we have

$$\begin{aligned}da^\alpha &= 4ia^\alpha|a|^2\theta - \phi_\beta{}^\alpha a^\beta - \tfrac{1}{2}\phi^\alpha \\ d\hat{a}^A &= 4i\hat{a}^A|\hat{a}|^2\hat{\theta} - \hat{\phi}_B{}^A\hat{a}^B - \tfrac{1}{2}\hat{\phi}^A\end{aligned} \tag{6.2}$$

Since the frames are adapted and the curve γ is contained in M we have $\hat{a}^\alpha = a^\alpha$ and $\hat{a}^b = 0$. Equating the corresponding terms in 6.2 we conclude

$$4ia^\alpha|a|^2\theta - \phi_\beta{}^\alpha a^\beta - \frac{1}{2}\phi^\alpha = 4ia^\alpha|a|^2\theta - \hat{\phi}_\beta{}^\alpha a^\beta - \frac{1}{2}\hat{\phi}^\alpha \tag{6.3}$$

12

Recall equation 4.4 which relates the pullback by f of the Chern-Moser forms on $\hat{M}$:

$$\hat{\phi}_\beta{}^\alpha = \phi_\beta{}^\alpha + C_\beta{}^\alpha\theta, \quad \hat{\phi}^\alpha = \phi^\alpha + C_\mu{}^\alpha\theta^\mu + F^\alpha\theta$$

Plugging these into equation 6.3 and canceling we then conclude, using 6.1, that along the curve γ

$$(2C_\beta{}^\alpha a^\beta + \frac{1}{2}F^\alpha)\theta = 0 \tag{6.4}$$

and since the curve is transversal to the complex tangent space (ie $\theta(\dot{\gamma}) \neq 0$) we then conclude

$$2C_\beta{}^\alpha a^\beta + \frac{1}{2}F^\alpha = 0 \tag{6.5}$$

In particular this holds when evaluated at the point $p \in M$. As we vary the chain through p, the values of $a = (a^\alpha)$ sweep out $\mathbb{C}^n$ and thus equation 6.5 holds as a polynomial of n complex variables a^α with constant coefficients (evaluated at p). Thus the preservation of chains implies $C_\beta{}^\alpha \equiv 0$ and $F^\alpha \equiv 0$. In view of equation 4.8 we have now established Theorem 6.1

7 Mappings Into Spheres

Now we assume $\hat{M}$ is (locally) the sphere $\mathbb{S}^{2\hat{n}+1} \subset \mathbb{C}^{\hat{n}+1}$ and thus its CR curvature tensor vanishes identically. $1 \Rightarrow 2$ in Theorem 0.1 holds by the previous section. To prove $2 \Leftrightarrow 3$ we observe by 4.1 that the map f may be lifted to a conformal isometry between Fefferman metrics if and only if:

$$\omega_\mu{}^a{}_\alpha\omega^\mu{}_{a\bar{\beta}} = \frac{1}{2(n+1)}\omega_\mu{}^a{}_\nu\omega^\mu{}_a{}^\nu g_{\alpha\bar{\beta}} \tag{7.1}$$

Lemma 7.1 *Condition 7.1 is satisfied if and only if $\omega_\alpha{}^a{}_\beta \equiv 0$*

The 'if' portion of the lemma is obvious. Now let us assume 7.1 holds. Fixing $\alpha = \beta$ we then obtain:

$$2(n+1)\sum_{\mu,\,a} |\omega_\mu{}^a{}_\alpha|^2 = \sum_{\mu,\,\nu,\,a} |\omega_\mu{}^a{}_\nu|^2$$

$$= \sum_{\mu,\,a} |\omega_\mu{}^a{}_\alpha|^2 + \sum_{\mu,\,\nu\neq\alpha,\,a} |\omega_\mu{}^a{}_\nu|^2$$

13

Which then implies for each α we have:

$$(2n+1) \sum_{\mu,\, a} |\omega_\mu{}^a{}_\alpha|^2 = \sum_{\mu,\, \nu \neq \alpha,\, a} |\omega_\mu{}^a{}_\nu|^2 \qquad (7.2)$$

Set $\xi_\alpha = \sum_{\mu,\, a} |\omega_\mu{}^a{}_\alpha|^2$ Summing over α equation 7.2 then implies:

$$(2n+1) \sum_\alpha \xi_\alpha = \sum_\alpha \sum_{\nu \neq \alpha} \xi_\nu = (n-1) \sum_\alpha \xi_\alpha \qquad (7.3)$$

Since $\xi_\alpha \geq 0$ we must have $\xi_\alpha = 0$ which then gives $\omega_\alpha{}^a{}_\beta \equiv 0$. QED

Now to prove $3 \Rightarrow 4$ in Theorem 0.1 we first recall that in [3] the following pseudoconformal Gauss equation is established for all $p \in M$:

$$[\hat{S}(X,Y,Z,V)] = S(X,Y,Z,V) + [\langle \Pi(X,Z), \Pi(Y,V) \rangle], \quad X,Y,Z,V \in T_p^{(1,0)}M \qquad (7.4)$$

where $[\hat{S}]$ denotes the traceless component of the pseudo-conformal curvature tensor $\hat{S}$ on $\hat{M}$. Since both $\hat{S}$ and Π vanish identically clearly so must S, that is to say that M is CR-flat and thus locally equivalent to the sphere. In [3] the following rigidity result is proved:

Theorem 7.1 *Let $f : M \hookrightarrow \mathbb{S}^{2\hat{n}+1}$ be a smooth CR-immersion and s be the degeneracy of f. If $\hat{n} - n - s \leq \frac{n}{2}$, then any other such CR-immersion $\tilde{f}$ is related to f by $\tilde{f} = A \circ f$, where A is a CR-automorphism of the ambient sphere.*

Since the CR second fundamental form of f vanishes it's degeneracy is $s = \hat{n} - n$. Let $\phi : \mathbb{S}^{2n+1} \to M$ be a local CR diffeomorphism guaranteed now since M is CR flat. The degeneracy of $f \circ \phi$ is the same as that of f and the CR second fundamental form of $f \circ \phi$ still vanishes, thus $f \circ \phi$ is equivalent to the trivial map, ie there is an automorphism $A \in Aut(\mathbb{S}^{2\hat{n}+1})$ such that the composition $A \circ f \circ \phi : \mathbb{S}^{2n+1} \to \mathbb{S}^{2\hat{n}+1}$ is the linear embedding.

Now we finish the proof of Theorem 0.1 by concluding $4 \Rightarrow 1$. Suppose maps $\phi : \mathbb{S}^{2n+1} \to M$ and $A \in Aut(\mathbb{S}^{2\hat{n}+1})$ exist as in the theorem so that $A \circ f \circ \phi : \mathbb{S}^{2n+1} \to \mathbb{S}^{2\hat{n}+1}$ is linear. It is known that the chains in a sphere are exactly the great circles and thus a linear embedding between spheres preserves chains. CR diffeomorphisms always preserve chains and thus since ϕ, A and $A \circ f \circ \phi$ all preserve chains, f must as well.

We have now established Theorem 0.1.

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